Representations of $U_q(\mathfrak{sl}_2)$ when q is a Root of Unity

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1 Classification of Irreducible Representations of $U_q(\mathfrak{sl}_2)$

Definition 1.1. *$U_q(\mathfrak{sl}_2)$ is the algebra over $\mathbb{C}$ generated by E, F, K, K^{-1} with the relations*

$$KK^{-1} = K^{-1}K = 1$$
$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

where $q \in \mathbb{C}^\times$ and $q \neq 1, -1$.

Lemma 1.2. *Let $[m] = \dfrac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + q^{m-3} + \ldots + q^{-(m-1)}$. Then*

$$[E, F^m] = [m]F^{m-1}\frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}}$$

Lemma 1.3. *Let V be a representation of $U_q(\mathfrak{sl}_2)$ and let $V^\lambda = \{v \in V \mid Kv = \lambda v\}$ be a weight space of V of weight λ. Then*

$$EV^\lambda \subset V^{q^2\lambda}, \quad FV^\lambda \subset V^{q^{-2}\lambda}$$

Before doing the case when q is a root of unity, let us first quickly review the generic case.

Theorem 1.4. *For q generic, let V be a f.d. irreducible representation of $U_q(\mathfrak{sl}_2)$ of dimension $n+1$. Then $V \cong L(\pm q^n)$.*

Proof. We first claim that any f.d. representation of $U_q(\mathfrak{sl}_2)$ has a highest weight vector. Since $\mathbb{C}$ is algebraically closed and V f.d, K has a nonzero eigenvector $Kw = \mu w$. If $Ew = 0$ we are done, otherwise consider the sequence $w, Ew, E^2w, \ldots$. By Lemma 1.3 it follows that they are all eigenvectors of K with distinct eigenvalues. Since V is f.d. it follows that $\exists k$ s.t. $E^kw \neq 0$ and $E^{k+1}w = 0$. and thus E^kw is a highest weight vector.

Thus let $v_0 \in V$ be a highest weight vector of weight λ. Consider the sequence $v_0, Fv_0, F^2v_0, \ldots, F^{n+1}v_0$. Again these are all eigenvectors for K with distinct eigenvalues. Since $\dim V = n+1$ one of them $F^jv_0 = 0$ for some j. But since V is irreducible it follows that $F^{n+1}v_0 = 0$ and $F^nv_0 \neq 0$ as otherwise we will pick up a submodule $\{v_0, \ldots, F^{j-1}v_0\}$. Now compute using Lemma 1.2

$$\begin{aligned} 0 = EF^{n+1}v_0 = [E, F^{n+1}]v_0 &= [n+1]F^n\frac{q^{-n}K - q^nK^{-1}}{q - q^{-1}}v_0 \\ &= [n+1]F^n\frac{q^{-n}\lambda - q^n\lambda^{-1}}{q - q^{-1}}v_0 \end{aligned}$$

Since $F^nV_0 \neq 0$ and $[n+1] \neq 0$ it follows that $q^{-n}\lambda = q^n\lambda^{-1} \implies \lambda = \pm q^n$. Thus we can define a nonzero morphism $V \to L(\pm q^n)$ by sending v_0 to v_λ and by simplicity of V and $L(\pm q^n)$, this will be an isomorphism. ∎

1.1 Representations when $q = e^{2\pi i/n}$

Now what goes wrong when $q = e^{2\pi i/n}$, a primitive $n-$th root of unity? Well two things, namely that since $q^n = 1$, the action of E and F on our sequences of vectors now may not have distinct eigenvalues. Also we can have $[d] = 0$ for certain values of d. Specifically,

Remark. Let $e = n$ when n is odd and $e = n/2$ when n is even, then $[e] = 0$. This is because when n is odd, the n terms $\left\{q^{n-1}, q^{n-3}, \ldots, q^{-(n-1)}\right\} = \left\{q^{-1}, \ldots, q^{-3}, \ldots, q^{n-n} = 1, q^{-2}, \ldots, q^{-(n-1)}\right\}$(where we have reduced the first half) are all the distinct $n-$th roots of unity, $x^n - 1$ so by Vieta the sum is zero. Similarly when $n = 2e$ is even, we have that $x^n - 1 = x^{2e} - 1 = (x^e + 1)(x^e - 1)$. $q^{e-1}, q^{e-3}, \ldots, q^{-(e-1)}$ will then correspond to either the e roots of $x^e + 1$ or $x^e - 1$ depending on whether e is even or odd and in both cases the sum will be zero. Note that now, the quantum numbers will be cyclic, for example when n is odd we have that

$$[n+1] = q^n + q^{-n-2} + \ldots + q^{-(n-2)} + q^{-n} = q\left(q^{n-1} + \ldots q^{-(n-1)}\right) + q^n = 0 + 1 = [1]$$

And similarly we have

$$[n+2] = q^{n+1} + q^{n-1} + \ldots + q^{-(n-1)} + q^{-(n+1)} = q^{n+1} + 0 + q^{-(n+1)} = q + q^{-1} = [2]$$

Thus it follows that $[n] = 0 \iff n \equiv 0 \pmod{e}$.

We start the classification of irreducibles L of $U_q(\mathfrak{sl}_2)$ for $q = e^{2\pi i/n}$ by splitting it up into three cases.

1.2 $\dim L < e$

Since $[\dim L] \neq 0$ as $\dim L < e$, and $\left\{q^{\dim L-1}, q^{\dim L-3}, \ldots, q^{-(\dim L-1)}\right\}$ are all distinct our method from the generic case applies and thus $L \cong L(\pm q^{\dim L-1})$.

1.3 $\dim L > e$

Lemma 1.5. *The elements E^e, F^e, K^e and the quantum Casimir $C_q = EF + \dfrac{qK^{-1} + q^{-1}K}{(q-q^{-1})^2}$ are in the center $\mathcal{Z}(U_q(\mathfrak{sl}_2))$ of $U_q(\mathfrak{sl}_2)$.*

Proof. By Lemma 1.2 we have that

$$[E, F^e] = [e]F^{e-1}\frac{q^{-(e-1)}K - q^{e-1}K^{-1}}{q - q^{-1}} = 0$$

Also, note that $KF^eK^{-1} = KK^{-1}q^{-2e}F^e = F^e$. You can compute for C_q. ■

Theorem 1.6. *There are no simple finite-dimensional $U_q(\mathfrak{sl}_2)$ modules of dimension $> e$.*

Proof. Let V be a $U_q(\mathfrak{sl}_2)-$module of dimension $> e$. We will show that V has a submodule of dimension $\leq e$.

Case 1: V has a lowest weight vector

Suppose $\exists v \in V$ s.t. $Fv = 0$ and $Kv = \alpha v$. Then we claim the subspace W generated by $v, Ev, \ldots, E^{e-1}v$ is a submodule. E^kv clearly stable under K and E when $k < e-1$. For $k = e-1$, note that

$$E(E^{e-1}v) = E^ev = c_1v$$

since $E^e \in \mathcal{Z}(U_q(\mathfrak{sl}_2))$ and V is f.d. so by Schur's lemma E^e acts by a scalar. To see that it's $F-$stable note by Lemma 1.2 we have

$$F(E^k v) = EFv + [F, E^k]v = [k]E^{k-1}\frac{q^{-(k-1)}K^{-1} - q^{k-1}K}{q - q^{-1}}v \in W$$

Case 2: V doesn't have a lowest weight vector

We now claim that the subspace W generated by $v, Fv, \ldots, F^{e-1}v$ is a submodule. Again $F^k v$ clearly stable under K, and also under F for $k < e - 1$. At $k = e - 1$, the same argument as above shows that $F^e v = c_2 v$, however we note that $c_2 \neq 0$ or otherwise one of the $F^k v$ will be a lowest weight vector. To check that W is $E-$stable, we don't have luxury of lowest or highest weight. Instead we use the Casimir.

$$E(F^k v) = EF(F^{k-1}v) = \left(C_q - \frac{qK^{-1} + q^{-1}K}{(q - q^{-1})^2}\right)(F^{k-1}v) = c_3 v + c_4 v \in W$$

When $k = 0$, note that $v = c_2^{-1}F^e v$. ■

1.4 $\dim L = e$

This is where all the interesting representations show up. There will be two infinite families of representations, essentially given by the two cases in the previous section. The first depends on two parameters λ and b.

Definition 1.7. *Let $\lambda \in \mathbb{C}^\times$ and let $M(\lambda)$ be the usual Verma module for $U_q(\mathfrak{sl}_2)$ and let $m_i = F^i m_0 = F^i \otimes 1$ where m_0 is the highest weight vector of weight λ. For $b \in \mathbb{C}$, consider the representation $Z_b(\lambda)$ of $U_q(\mathfrak{sl}_2)$ given by*

$$Z_b(\lambda) = M(\lambda)/U_q(\mathfrak{sl}_2)(bm_0 - m_e)$$

Since

$$Em_e = EF^e m_0 = [E, F^e]m_0 = [e]\ldots = 0 \implies E(bm_0 - m_e) = 0$$

and $K(bm_0 - m_e) = \lambda(bm_0 - m_e)$ it follows that $U_q(\mathfrak{sl}_2)(bm_0 - m_e)$ is spanned by $F^i(bm_0 - m_e) = bm_i - m_{e+i}$[1] *and therefore $Z_b(\lambda)$ has basis $m_0, m_1, \ldots, m_{e-1}$ and the action of K, F, E on this basis is given by*

$$\begin{aligned}
Km_i &= q^{-2i}\lambda m_i \\
Fm_i &= \begin{cases} m_{i+1} & \text{if } i < e-1 \\ bm_0 & \text{if } i = e-1 \end{cases} \\
Em_i &= \begin{cases} 0 & \text{if } i = 0 \\ [i]\dfrac{q^{-(i-1)}\lambda - q^{i-1}\lambda^{-1}}{q - q^{-1}}m_{i-1} & \text{if } i > 0 \end{cases}
\end{aligned}$$

Remark. For $b = 0$, $\lambda = q^{e-1}$, we exactly get the $e-$dimensional module $L(q^{e-1})$ so the other representations at least for $b = 0$ are like a one-parameter deformation of $L(q^{e-1})$ where the highest weight λ can be anything you want now instead of being constrained to q^{e-1}

Lemma 1.8. *$Z_b(\lambda)$ is irreducible if and only if $b \neq 0$ or if $b = 0$ and $\lambda \neq \pm q^k$ where $0 \leq k \leq e - 2$.*

The second family of representations will depend on three parameters λ, a, b.

[1] i.e. these are the only relations between the basis vectors.

Definition 1.9. *$L(\lambda, a, b)$ is the representation of $U_q(\mathfrak{sl}_2)$ given by*

$$Km_i = q^{-2i}\lambda m_i$$
$$Fm_i = \begin{cases} m_{i+1} & \textit{if } i < e-1 \\ bm_0 & \textit{if } i = e-1 \end{cases}$$
$$Em_i = \begin{cases} am_{e-1} & \textit{if } i = 0 \\ abm_{i-1} + [i]\dfrac{q^{-(i-1)}\lambda - q^{i-1}\lambda^{-1}}{q-q^{-1}}m_{i-1} & \textit{if } i > 0 \end{cases}$$

Remark. When $a = 0$, we recover $Z_b(\lambda)$, so $L(\lambda, a, b)$ can be thought of as a one-parameter deformation of $Z_b(\lambda)$.

Lemma 1.10. *$L(\lambda, a, b)$ is irreducible*[2] $\iff$ *$Z_b(\lambda)$ is irreducible, aka a doesn't affect simplicity.*

Fortunately, this one parameter deformation business stops here as we have

Theorem 1.11. *Any simple $U_q(\mathfrak{sl}_2)-$module M of dimension e is isomorphic to a module in the following list*

(i) $Z_b(\lambda)$ where $b \neq 0$ or $b = 0$ and $\lambda \neq \pm q^k$ where $0 \leq k \leq e-2$.

(ii) $Z_b(\lambda)^\omega$ where $b \neq 0$ or $b = 0$ and $\lambda \neq \pm q^k$ where $0 \leq k \leq e-2$

(iii) $L(\lambda, a, b)$ where $a, b \neq 0$.

Proof. Case 1: E^e acts by 0 on M. This implies that the $0-$eigenspace of E, $E_0 = \{m \in M | Em = 0\}$ is nonempty or otherwise $E^e m = 0 \implies m = 0$. As E_0 is $K-$stable, this means we have an eigenvector $v_0 \in E_0$ with eigenvalue λ for the action of K, aka v_0 is a h.w. vector of weight λ. By the universal property of $M(\lambda)$, it follows that we have a $U_q(\mathfrak{sl}_2)-$linear map $\varphi : M(\lambda) \to M$ sending m_0 to v_0. Let b be the scalar that F^e acts by M on. Then

$$\varphi(bm_0 - m_e) = \varphi(bm_0 - F^e m_0) = bv_0 - F^e v_0 = bv_0 - bv_0 = 0$$

Thus φ factors through $Z_b(\lambda) \to M$ and since both sides are simple, this is an isomorphism.
Case 2: F^e acts by 0 on M. Use the Cartan involution of $U_q(\mathfrak{sl}_2)$ which is an algebra isomorphism of $U_q(\mathfrak{sl}_2)$ which sends

$$\omega(E) = F, \quad \omega(F) = E, \quad \omega(K) = K^{-1}$$

Aka notice that now E^e acts by zero on M^ω where M^ω as a set equals M but the action of $x \in U_q(\mathfrak{sl}_2)$ on M^ω is given by $\omega(x) \cdot m$ instead. It follows that $M^\omega \cong Z_b(\lambda) \implies M \cong Z_b(\lambda)^\omega$.
Case 3: Both E^e and F^e do not act by 0 on M Apply same analysis but now you see how the Casimir acts, and then we end up with $L(\lambda, a, b)$. ■

So we have just classified all finite-dimensional simples of $U_q(\mathfrak{sl}_2)$. However it turns out that we actually classified all simples of $U_q(\mathfrak{sl}_2)$ because

Proposition 1.12. *Any simple $U_q(\mathfrak{sl}_2)-$module V is finite-dimensional.*

[2] $L(\lambda, a, b)$ is always indecomposable, however.

Proof. Notice that since $\mathcal{Z}(U_q(\mathfrak{sl}_2))$ is very big aka contains E^e, F^e, K^e, we will have that $U_q(\mathfrak{sl}_2)$ is a f.g. $\mathcal{Z}(U_q(\mathfrak{sl}_2))-$module by PBW, spanned by $F^iK^kE^j$ where $i,j,k < e$. Therefore let $U_q(\mathfrak{sl}_2) = \sum_{i=1}^{r} \mathcal{Z}(U_q(\mathfrak{sl}_2))u_i$, then for V simple we have that

$$V = U_q(\mathfrak{sl}_2)v = \sum_{i=1}^{r} \mathcal{Z}(U_q(\mathfrak{sl}_2))u_i \cdot v$$

This shows that V is a f.g. $\mathcal{Z}(U_q(\mathfrak{sl}_2))$ module which is a Noetherian ring (as it's a f.g. algebra over a field $\mathbb{C}$ so by Hilbert's Basis theorem $\mathbb{C}[x_1, \ldots, x_n]$ is a Noetherian ring and thus any quotient will also be a Noetherian ring) and thus V is a Noetherian $\mathcal{Z}(U_q(\mathfrak{sl}_2))-$module. As a result, there is a maximal $\mathcal{Z}(U_q(\mathfrak{sl}_2))$ proper submodule of V, namely V'. It follows that V/V' is a simple $\mathcal{Z}(U_q(\mathfrak{sl}_2))-$module and therefore we have an isomorphism of $\mathcal{Z}(U_q(\mathfrak{sl}_2))-$modules

$$\mathcal{Z}(U_q(\mathfrak{sl}_2))/\mathfrak{m} \cong V/V'$$

for some maximal ideal $\mathfrak{m}$ in $\mathcal{Z}(U_q(\mathfrak{sl}_2))$. Thus, $\mathfrak{m}V \subset V'$ so $\mathfrak{m}V$ is a strict subset of V. But since $\mathfrak{m}$ is in the center, it's also a $U_q(\mathfrak{sl}_2)-$module and so by simplicity of V it follows that $\mathfrak{m}V = 0$. Hence the action of $\mathcal{Z}(U_q(\mathfrak{sl}_2))$ factors through $\mathcal{Z}(U_q(\mathfrak{sl}_2))/\mathfrak{m} = \mathbb{C}$ and so r is actually a bound on the dimension of V and thus V is finite-dimensional. ■

Let us contrast with the generic case, where all of our irreducibles were highest weight representations and we could have irreducibles that were infinite-dimensional, i.e. the Vermas $M(\lambda)$ for which $\lambda \neq \pm q^n$. For q a root of unity, most of the irreducibles are not highest weight representations and we can't have infinite-dimensional representations. In fact, most of the irreducibles are of dimension e, so these are in a sense, dense in the set of all irreducibles. We can actually make this precise.

1.5 Representations of $\mathfrak{sl}_2$ when $\operatorname{char} k = p$

Here is a brief interlude on the representation theory of $\mathfrak{sl}_2$ when k is an algebraically closed field of $\operatorname{char} k = p$. We will have a big center in $U(\mathfrak{sl}_2)$, namely because $\mathfrak{sl}_2$ over a field of characteristic p has the structure of a restricted lie algebra, namely a lie algbera with a $p-$power map $x \mapsto x^{[p]}$ that satisfies nice properties.

Lemma 1.13. *Let A be an associative algebra over a field $\mathbb{F}$ of characteristic p, and let $x, y \in A$. Let* $\operatorname{ad}(x)(y) := xy - yx$ *for $x, y \in A$. Then* $\operatorname{ad}(x)^p(y) = [x^p, y]$

Proof. Write $\operatorname{ad}(x) = \ell_x - r_x$ and use binomial theorem. ■

Proposition 1.14. *For $x \in \mathfrak{sl}_2$ we have $\xi(x) = x^p - x^{[p]} \in \mathcal{Z}(U(\mathfrak{sl}_2))$.*

Proof. First note that $x^p = x \otimes x \ldots \otimes x \in U(\mathfrak{sl}_2)$ and I will define $x^{[p]}$ to be the $p-$th power of x in the associative algebra $\mathfrak{gl}_2$. Applying lemma twice, once in $U(\mathfrak{sl}_2)$ to x^p and then in $\mathfrak{sl}_2$ to $x^{[p]}$ yields that $[x^p, y] = \operatorname{ad}(x)^p(y) = [x^{[p]}, y]$ for all $y \in \mathfrak{sl}_2$ which generates $U(\mathfrak{sl}_2)$ so this yields the result. ■

Just like in $U_q(\mathfrak{sl}_2)$, one can show that $U(\mathfrak{sl}_2)$ is a f.g. module over $\mathcal{Z}(U(\mathfrak{sl}_2))$ and this implies that all simples of $\mathfrak{sl}_2$ are finite-dimensional. Also like $U_q(\mathfrak{sl}_2)$ the action of the central elements will completely determine the representation theory. On simple modules M, $\xi(x)$ must act by a scalar $\chi_M(x)^p$ for all $x \in \mathfrak{sl}_2$. We now define the "baby" Verma modules

Definition 1.15. *Let $\lambda \in k^\times$, and $\chi \in \mathfrak{sl}_2^*$, then $Z_\chi(\lambda)$ is the $\mathfrak{sl}_2-$module with basis $\{v_i = f^i \otimes m_0 \,|\, 0 \le i < p\}$ where the action of h, e, f is given by*

$$hv_i = (\lambda - 2i)\lambda v_i$$
$$fv_i = \begin{cases} v_{i+1} & \text{if } i < p-1 \\ \chi(f)^p v_0 & \text{if } i = p-1 \end{cases}$$
$$ev_i = \begin{cases} 0 & \text{if } i = 0 \\ i(\lambda - (i-1))v_{i-1} & \text{if } i > 0 \end{cases}$$

Theorem 1.16. *Any irreducible $\mathfrak{sl}_2-$module*[3] *over k is isomorphic to a module in the following list.*

(i) $L(\lambda)$ where $\lambda \in k$ is an integer $0 \le \lambda < p$.

(ii) $Z_\chi(\lambda)$ where $\chi \neq 0$ and $\lambda \notin \mathbb{F}_p$ (semisimple).

(iii) $Z_\chi(\lambda)$ where $\chi(f) = 1$ and $\lambda \in \mathbb{F}_p$ (nilpotent).

Like before most of the irreducibles will have dimension p. However, we don't have the analogue of $L(\lambda, a, b)$ here. The discrepancy is because $\text{Spec}\mathcal{Z}^0(U(\mathfrak{sl}_2)) = \text{Spec}\overline{\mathbb{F}}[x, y, h] \cong \overline{\mathbb{F}}^3$ while $\text{Spec}\mathcal{Z}^0(U_q(\mathfrak{sl}_2)) = \text{Spec}\mathbb{C}[E, F, K, K^{-1}] = \mathbb{C}^2 \times \mathbb{C}^\times$ so we need two variables for the $\mathbb{C}^\times$ term.

2 Representations for general $U_q(\mathfrak{g})$

I won't define $U_q(\mathfrak{g})$ for general simple lie algebras $\mathfrak{g}$, but you can use your imagination. Now it turns out there's multiple ways to deal with quantum groups at roots of unity with different representation theories and what I just explained was just one form. Namely consider $U_q(\mathfrak{g})$ as an algebra over $\mathbb{C}(q)$ now and let $A = \mathbb{C}[q, q^{-1}]$. A $A-$subalgebra U_A of $U_q(\mathfrak{g})$ is an integral form if $U_A \otimes_A \mathbb{C}(q) \cong U_q(\mathfrak{g})$. The corresponding quantum group will then be

$$U_\epsilon = U_A \otimes_A \mathbb{C}$$

where $q \in A$ acts on $\mathbb{C}$ by ϵ. The integral form we were working with is called the Kac-De Concini form where technically we have to replace $[E, F]$ with a formal symbol, but the algebra is literally the same. For this integral form we have

Theorem 2.1. *Let $\Phi : \text{Rep}^{irr}(U_q(\mathfrak{g})) \to \text{Spec}Z_\epsilon = \mathcal{Z}(U_q(\mathfrak{g}))$ be the map sending an irreducible $U_q(\mathfrak{g})$ to it's central character. Then there is a nonzero proper subvariety*[4] *$\mathcal{D}$ of $\text{Spec}\mathcal{Z}_\epsilon$ such that*

(i) If $\chi \in \mathcal{Z}_\epsilon \setminus \mathcal{D}$ then $\Phi^{-1}(\chi)$ consists of a single irreducible $U_q(\mathfrak{g})-$module of dimension e^N where N is the number of positive roots.

(ii) If $\chi \in \mathcal{D}$, then $\Phi^{-1}(\chi)$ consists of a finite number of irreducible $U_q(\mathfrak{g})$ modules of dimension $< e^N$

This integral form should correspond with representations of $\mathfrak{sl}_2$ in characteristic p, but this is a very hard problem. However if we ask the analogous question for algebraic groups, it turns out there's much more progress.

[3]Technically we have fixed a central character already...

[4]I mean actual variety here, aka closed points of the affine scheme.

2.1 Restricted Quantum Group

Definition 2.2. *Let $E_i^{(k)} = E_i^k/[k]!$ and $F_i^{(k)} = F_i^k/[k]!$ and then consider the subalgebra of $U_q(\mathfrak{g})$ generated by $E_i^{(k)}, F_i^{(k)}, K_i^{\pm 1}$ where $k \geq 0$, and $i = 1, \dots, n$. Then the restricted quantum group $U_q^{res}(\mathfrak{g})$ will be the $\mathbb{Z}[q, q^{-1}]$ algebra generated by the formal symbols $E_i^{(k)}, F_i^{(k)}, K_i^{\pm 1}$ and relations given by the relations that would hold in $U_q(\mathfrak{g})$.*

Definition 2.3. *Let $q = \zeta \in \mathbb{C}^\times$. Define the specialization of the restricted quantum group to be*

$$U_\zeta^{res}(\mathfrak{g}) := U_q^{res}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$$

where we let $q \to \zeta$.

Warning. When ζ is a $\ell-$th root of unity, $U_\zeta^{res}(\mathfrak{g})$ can have weird relations. For simplicity, assume that ℓ is odd. Then note that $E_i^\ell = [\ell]! E_i^{(\ell)} = 0$ and similarly for F_i^k. In fact, we also have that $K_i^{2\ell} = 1$. To see this, first in $U_q(\mathfrak{g})$ define

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i q^{c-s+1} - K_i^{-1} q^{-c+s-1}}{q^s - v^{-s}}$$

It is a theorem of Lusztig that $\begin{bmatrix} K_i; c \\ t \end{bmatrix} \in U_q^{res}(\mathfrak{g})$ (Essentially comes from the commutator $[E_i^{(c)}, F_i^{(t)}]$). Thus in the specialization for $c = 0$ and $t = \ell$ we see that

$$\prod_{s=1}^{\ell} K_i \zeta^{-s+1} - K_i^{-1} \zeta^{s-1} = \begin{bmatrix} K_i; 0 \\ \ell \end{bmatrix} \prod_{s=1}^{\ell} \zeta^s - \zeta^{-s} = 0 \in U_\zeta^{res}(\mathfrak{g})$$

Now multiply the LHS by $K_i^\ell \prod_{s=1}^{\ell} \zeta^{(s-1)}$ and we obtain

$$\prod_{s=1}^{\ell} K_i^2 - (\zeta^2)^{s-1} = 0$$

And now the LHS is precisely $P(K_i)$ where $P(x)$ is defined via $x^{2\ell} - 1 = (x^2 - 1)P(x)$. Thus $K_i^{2\ell} = 1$ as desired.

Remark. Finite-dimensional irreducibles of $U_q^{res}(\mathfrak{g})$ turn out to be parameterized by highest dominant weight exactly like the classical situation. However the structure of the irreducible representations $L(\lambda)$ are different from the usual $L(\lambda)$ in the classical setting. There is also a connection to modular representation theory of $\mathfrak{g}$ but not in the sense of lie algebra but in the sense of algebraic group. Namely it turns out that the study of $L(\lambda)$ and its characters reduce to a finite set, namely the set of $p-$restricted weights of $\mathfrak{X}$, where are exactly the weights that can be written in the form

$$\lambda = \sum \lambda_\alpha \omega_\alpha, \quad 0 \leq \lambda_\alpha < p$$

where ω_α are the fundamental weights. Then we have a Steinberg tensor product theorem

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \ldots \otimes L(\lambda_m)^{(m)}$$

where (k) is the $k-$th Frobenius twist.

So $U_q^{res}(\mathfrak{g})$ is a characteristic 0 object that is exhibiting characteristic p behavior, so one might wonder if one can use this to prove things using tools from the other world? The answer is Yes, namely the analogue of the KL conjectures for algbebraic groups in characteristic p

Lusztig's Conjecture: Let G be an algebraic group with lie algebra $\mathfrak{g}$ over an algebraically closed field k of characteristic p. Suppose $p \geq h$ Then

$$\mathrm{ch}\,(L_x) = \sum_{y \in W^f} (-1)^{\ell(x)+\ell(y)} P_{w_0 y, w_0 x}(1) \mathrm{ch}(\nabla_y)$$

The L_x are f.d. now vs the original KL conjectures where the $L(\lambda)$ are infinite-dimensional.
Lusztig's Program then involves proving

1. Show that the process of "reduction modulo p" from representations of $U_q^{res}(\mathfrak{g})$ for q a root of unity to representations of G in characteristic p takes irreducibles to irreducibles.
2. Show that representations of $U_q^{res}(\mathfrak{g})$ for q a root of unity are closed related to representations of affine lie algebra $\widehat{\mathfrak{g}}$ with central charge $-p-h$. item Show that characters of irreducible h.w. representations of $\widehat{\mathfrak{g}}$ can be related to intersection cohomology of Schubert varieties in an affine flag variety.
3. Show that IC is computed by the polynomials $P_{y,w}$.

Step 1 was solved in the compliment of a finite set, aka for $p >>>>> 0$. Step 2 through 4 well known in the 1990-2000's. Is there a way to drop the condition on p though? Well, no, due to recent work of Williamson Lusztig's conjecture is false!!.